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Stability of functional equations in single variable ☆

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Abstract

This paper discusses Hyers–Ulam stability for functional equations in single variable, including the forms of linear functional equation, nonlinear functional equation and iterative equation. Surveying many known and related results, we clarify the relations between Hyers–Ulam stability and other senses of stability such as iterative stability, continuous dependence and robust stability, which are used for functional equations. Applying results of nonlinear functional equations we give the Hyers–Ulam stability of Böttcher's equation. We also prove a general result of Hyers–Ulam stability for iterative equations.

Keywords: Functional equation; Iteration; Hyers–Ulam stability; Iterative stability; Convergence

1. Introduction

Hyers–Ulam stability is a basic sense of stability for functional equations. Usually the functional equation

$$E_1(\varphi) = E_2(\varphi) \tag{1.1}$$

is said to have the *Hyers–Ulam stability* if for an approximate solution φ_s such that

$$\left|E_1(\varphi_s)(x) - E_2(\varphi_s)(x)\right| \leqslant \delta \tag{1.2}$$

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for some fixed constant $\delta \geqslant 0$ there exists a solution φ of Eq. (1.1) such that

$$\left|\varphi(x) - \varphi_s(x)\right| \leqslant \varepsilon \tag{1.3}$$

for some positive constant ε. Sometimes we call φ_s a *δ-approximate solution* of Eq. (1.1) and φ is *ε-close* to φ_s.

Such an idea of stability was given in 1940 by Ulam [30] for Cauchy equation

$$\varphi(x + y) = \varphi(x) + \varphi(y)$$

and his problem was solved by Hyers [9] in 1941. Later, the Hyers–Ulam stability was studied extensively (see, e.g., [2,20,23,26]). This concept is also generalized in [7,8,16,24]. As in [17] we say Eq. (1.1) has the *generalized Hyers–Ulam–Rassias stability* if for an approximate solution φ_s such that

$$\left|E_1(\varphi_s)(x) - E_2(\varphi_s)(x)\right| \leqslant \psi(x) \tag{1.4}$$

for some fixed function $\psi(x)$ there exists a solution φ of Eq. (1.1) such that

$$\left|\varphi(x) - \varphi_s(x)\right| \leqslant \Phi(x) \tag{1.5}$$

for some fixed function $\Phi(x)$. We say Eq. (1.1) has the *stability in the sense of Ger* if for an approximate solution φ_s such that

$$\left|\frac{E_1(\varphi_s)(x)}{E_2(\varphi_s)(x)} - 1\right| \leqslant \psi(x) \tag{1.6}$$

for some fixed function $\psi(x)$ there exists a solution φ of Eq. (1.1) such that

$$\alpha(x) \leqslant \frac{\varphi(x)}{\varphi_s(x)} \leqslant \beta(x) \tag{1.7}$$

for some fixed functions $\alpha(x)$ and $\beta(x)$.

Most of known results on stability of functional equations are given for those equations in several variables. Hyers surveyed the stability of isometries in [10] and later, jointly with Rassias, the stability of homomorphisms in [11]. Another survey on Hyers–Ulam stability of functional equations in several variables is given by Forti [6]. In contrast, there are much less results of stability for functional equations in single variable and so far no surveys on single variable ones are found. To promote investigation in stability of functional equations in single variable, this paper provides such a survey, including linear functional equations, nonlinear functional equations and iterative equations. We clarify the relations between Hyers–Ulam stability and other senses of stability such as iterative stability, continuous dependence and robust stability, which are used for functional equations. Applying results of nonlinear functional equations we give the Hyers–Ulam stability of Böttcher's equation. Hyers–Ulam stability is also discussed for a general form of iterative equations which include the polynomial-like iterative equation with variable coefficients.

2. Linear equations: iterative stability

Of the most importance is the linear functional equation (see [18])

$$\varphi\bigl(f(x)\bigr) = g(x)\varphi(x) + h(x), \tag{2.8}$$

where f, g, h are given functions on an interval I and φ is unknown. When $h(x) \equiv 0$ this equation, i.e.,

$$\varphi\bigl(f(x)\bigr) = g(x)\varphi(x), \tag{2.9}$$

is called *homogeneous linear equation*.

In 1970 Brydak [3] introduced a concept of stability for the linear equation (2.8), which was referred to as iterative stability later by Turdza [28]. Not working at Eq. (2.8) directly, he discussed its equivalent form

$$\varphi\bigl(f^n(x)\bigr) = G_n(x)\varphi(x) - G_n(x)\sum_{i=0}^{n-1} h\bigl(f^i(x)\bigr)/G_{i+1}(x), \quad n = 1, 2, \ldots,$$

where $f^i(x)$ is the ith iterate of f and

$$G_n(x) := \prod_{i=0}^{n-1} g\bigl(f^i(x)\bigr), \quad \forall x \in I, \tag{2.10}$$

as given in [18]. Equation (2.8) is said to be *iteratively stable* on the interval I with respect to a class $\mathcal{C}$ of functions if there exists a constant $K > 0$ such that for each positive number ε and each function $\varphi_s \in \mathcal{C}$ which satisfies the inequalities

$$\left|\varphi_s\bigl(f^n(x)\bigr) - G_n(x)\varphi_s(x) - G_n(x)\sum_{i=0}^{n-1} h\bigl(f^i(x)\bigr)/G_{i+1}(x)\right| \leqslant \varepsilon, \quad n = 1, 2, \ldots, \tag{2.11}$$

for all $x \in I$ there exists a solution $\varphi \in \mathcal{C}$ of (2.8) on I such that $|\varphi(x) - \varphi_s(x)| \leqslant K\varepsilon$ for all $x \in I$. Obviously, the first inequality in (2.11) (i.e., the inequality of $n = 1$) is what the Hyers–Ulam stability requires for (2.8). Therefore, iteratively stability for Eq. (2.8) can be regarded as a weak notion of Hyers–Ulam stability.

Consider (2.8) on I, which can be any finite or infinite interval. In the case that I is infinite we require, instead of continuity of solutions at $-\infty$ (or ∞), the existence of a finite limit of the solutions under discussion at $-\infty$ (or ∞). Suppose

(H1) g, h are continuous on an interval I and $g(x) \neq 0$ for all $x \in I$;
(H2) f is strictly increasing and continuous on I, and there exists a point $\xi \in I$ such that $(f(x) - x)(\xi - x) > 0$ and $(f(x) - \xi)(\xi - x) < 0$ for all $x \in I \backslash \{\xi\}$ (i.e., ξ is a unique attractive fixed point of f).

Theorem 1 [3]. *Suppose that* (H1) *and* (H2) *hold and that Eq.* (2.8) *has a continuous solution. If the case*

(A) *The limit $G(x) = \lim_{n\to\infty} G_n(x)$ exists and is continuous for all $x \in I$, and $G(x) \neq 0$ for $x \in I$*

occurs and if $\sup_{x\in I} 1/|G(x)| < \infty$, *then Eq.* (2.8) *is iteratively stable in* I *with respect to the class* $\mathcal{C}(I)$ *of continuous functions.*

Theorem 2 [3]. *Suppose that* (H1) *and* (H2) *hold and the case*

(B) *There exists an interval* $J \subset I$ *such that* $G_n(x) \to 0$ *uniformly in* J *as* $n \to \infty$

occurs. If there exists a point $x_0 \in I$ *such that the interval* $I_0 = [f(x_0), x_0]$ (*or* $I_0 = [x_0, f(x_0)]$) *is contained in* J, *then Eq.* (2.8) *is iteratively stable in the interval* $[\xi, x_0]$ (*or in the interval* $[x_0, \xi]$), *provided that Eq.* (2.8) *has a continuous solution in* I.

As $f(x_0) < x_0$, only the interval $[\xi, x_0]$, which on the left-hand side of x_0, is discussed in Brydak's theorem of "iteratively stable." The right-hand side of x_0 is investigated further by Turdza [28]. Consider Eq. (2.8) on $I = (a, b)$, where a, b may take $-\infty$ or ∞. If $a = -\infty$, for example, functions f, g, h are assumed additionally to be convergent $x \to -\infty$. In his definition of iterative stability, condition (2.11) is replaced with the inequalities

$$\left|\varphi_s\big(f^n(x)\big) - S^n\varphi_s\big(f^n(x)\big)\right| \leqslant \varepsilon, \quad n = 1, 2, \ldots, \ \forall x \in I, \tag{2.12}$$

where $S\varphi(x) := g(f^{-1}(x))\varphi(f^{-1}(x)) + h(f^{-1}(x))$. Actually, his definition is equivalent to Brydak's because we can prove inductively that

$$S^n\varphi\big(f^n(x)\big) = G_n(x)\varphi(x) + G_n(x)\sum_{i=0}^{n-1}\frac{h(f^i(x))}{G_{i+1}(x)}, \quad n = 1, 2, \ldots.$$

Under the same hypotheses (H1) and (H2) he proved the following result in case (B).

Theorem 3. *Equation* (2.8) *is iteratively stable in every interval* $[x_0, b)$ ($\xi < x_0 < b$) *if either* (i) $f^r(b) < x_0$ *for some integer* $r > 0$, *or* (ii) $g(x) \geqslant 1$ *in a neighborhood* $(x_1, b]$ *of* b.

Furthermore, Choczewski et al. [5] obtained the following results under the hypothesis

(H) $I = |0, A|$, where $0 < A \leqslant \infty$ and $|0, A|$ stands for any of the four intervals with the ends 0 and A. $f : I \to I$ is a continuous and strictly increasing function and $0 < f(x) < x$ in $I\backslash\{0\}$. $g : I \to \mathbf{R}$ is continuous in I, $g(x) \neq 0$ in $I\backslash\{0\}$. $h : I \to \mathbf{R}$ is also continuous in I.

Theorem 4. *Suppose that* (H) *holds and that there exists a continuous solution of* (2.8) *on* I. *If Eq.* (2.9) *is Hyers–Ulam stable* (*respectively, iteratively stable*) *on* I, *then the same is true for Eq.* (2.8).

Let (H_0) stand for (H) in the case that $h = 0$. In case (B) there always exists an open set, say $U \subset I$, which is maximal for almost uniform convergence of G_n to zero.

Theorem 5. *Assume that* (H_0) *and case* (B) *hold. If there exist positive numbers* C_1, C_2, C_3*, an interval* $I_0 := [f(x_0), x_0] \subset I \backslash \{0\}$ *and a point* $x_1 \geqslant x_0$ *such that for every positive integer* n *we have* $C_1 \leqslant |G_n(x)|$ *in* $I \backslash (U \cup 0)$, $|G_n(x)| \leqslant C_2$ *in* $I_0 \cap U$ *and* $|G_n(f^{-n}(x))| \geqslant C_3$ *for* $x \in [x_1, A]$*, then Eq.* (2.9) *is iteratively stable in* I.

Theorem 4 gives a relation between the inhomogeneous equation (2.8) and the homogeneous (2.9) on their stability. Theorem 4 together with Theorem 5 tells when Eq. (2.8) is iteratively stable in the class $C(I)$.

Turdza also generalizes those results to Banach spaces. Suppose

(H′) I is the interval $[0, b)$, where $0 < b \leqslant \infty$, K is the field of real or complex numbers, Y is a Banach space over K, the functions $f : I \to I$, $g : I \to K$ and $h : I \to Y$ are continuous, and $\varphi : I \to Y$. Furthermore, $0 < f(x) < x$ for $x \in (0, b)$, $f(0) = 0$, f is strictly increasing in I, $g(x) \neq 0$ for $x \in I$.

For the sequence G_n defined in (2.10), there are actually three possible cases:

(A′) There is a continuous function $D : I \to K$, $D(x) \neq 0$ for $x \in I$, such that $\lim_{n\to\infty} G_n(x) = D(x)$ for $x \in I$;
(B′) There is an interval $J \subset I$ such that $\lim_{n\to\infty} G_n(x) = 0$ uniformly in J;
(C′) Neither (A′) nor (B′) is fulfilled.

In [29] Turdza proved the following

Theorem 6. *Assume that hypothesis* (H′) *is fulfilled and that case* (A′) *occurs. If there exists a continuous solution of Eq.* (2.8) *on* I *and the function* $1/D(x)$ *is bounded, then Eq.* (2.8) *is iteratively stable in* I *in the class of functions continuous on* I*. If moreover the series* $\lim_{n\to\infty} \sum_{i=1}^{n} |G_n(x)/G_i(x)|$ *is bounded, then Eq.* (2.8) *is Hyers–Ulam stable in the class of functions continuous on* I.

The result on Hyers–Ulam stability in this theorem is not complete. In fact, since $D(x) \neq 0$ and both $1/D(x)$ and $\lim_{n\to\infty} \sum_{i=1}^{n} |G_n(x)/G_i(x)|$ are bounded, we have

$$\lim_{n\to\infty} \sum_{i=1}^{n} \frac{1}{|G_i(x)|} = \lim_{n\to\infty} \frac{\sum_{i=1}^{n} |G_n(x)|/|G_i(x)|}{|G_n(x)|}$$
$$= \frac{\lim_{n\to\infty} \sum_{i=1}^{n} |G_n(x)|/|G_i(x)|}{|D(x)|},$$

implying that $\sum_{i=1}^{\infty} 1/|G_i(x)|$ converges. It follows that $\lim_{i\to\infty} 1/|G_i(x)| = 0$, or equivalently that $\lim_{i\to\infty} G_i(x) = \infty$, a contradiction to the assumption of case (A′). This implies that the assumptions in Theorem 6 are incompatible.

3. Linear equations: Hyers–Ulam stability

In 1991 Baker [1] discussed Hyers–Ulam stability for linear equations of the form

$$\varphi(x) = g(x)\varphi\big(f(x)\big) + h(x). \tag{3.13}$$

His result is proved by applying the contraction principle.

Theorem 7. *Let S be a nonempty set and E be a real* (*or complex*) *Banach space. Consider $f : S \to S$, $h : S \to E$, $g : S \to \mathbf{R}$* (*or* $\mathbf{C}$) *with $|g(x)| \leqslant \lambda$ for all $x \in S$ and $0 \leqslant \lambda < 1$. Suppose that $\varphi_s : S \to E$ satisfies*

$$\big\|\varphi_s(x) - g(x)\varphi_s\big(f(x)\big) - h(x)\big\| \leqslant \delta \quad \textit{for all } x \in S, \tag{3.14}$$

where $\delta > 0$ is a constant. Then there exists a unique function $\varphi : S \to E$ which satisfies Eq. (3.13) *and*

$$\big\|\varphi(x) - \varphi_s(x)\big\| \leqslant \delta/(1-\lambda), \quad \forall x \in S. \tag{3.15}$$

Moreover, he also proved a similar result on Hyers–Ulam stability for Eq. (3.13) when E is a Banach algebra.

Theorem 7 is applicable to linear equations of the form (2.8) if $g(x) \neq 0$, but it requires either that $\|g(x)\| \geqslant r$ for all $x \in S$ for a certain constant $r > 1$, as reduced to

$$\varphi(x) = \big(g(x)\big)^{-1}\varphi\big(f(x)\big) - \big(g(x)\big)^{-1}h(x), \tag{3.16}$$

or that $f : S \to S$ is homeomorphic and $|g(x)| \leqslant \lambda$ for all $x \in S$ for a certain constant $0 \leqslant \lambda < 1$, as reduced to

$$\varphi(x) = g\big(f^{-1}(x)\big)\varphi\big(f^{-1}(x)\big) + h\big(f^{-1}(x)\big). \tag{3.17}$$

For homogeneous equation (2.9) in some special forms, without those requirements for (3.16) and (3.17), results on Hyers–Ulam stability are obtained by constructing convergent sequences. More concretely, the Hyers–Ulam stability and the generalized Hyers–Ulam–Rassias stability for equation

$$\varphi(x+p) = k\varphi(x) \tag{3.18}$$

were discussed by Lee and Jun [19]. The three senses of the Hyers–Ulam stability mentioned in (1.2)–(1.7) were discussed by Jung [12–14] for the gamma functional equation

$$\varphi(x+1) = x\varphi(x), \quad x \in (0, +\infty). \tag{3.19}$$

Later Kim [15] generalized Jung's results to the generalized gamma functional equation

$$\varphi(x+p) = g(x)\varphi(x) \tag{3.20}$$

and obtained the following results.

Theorem 8. *If function $\varphi_s : (0,\infty) \to \mathbf{R}$ satisfies the inequality $|\varphi_s(x+p) - g(x)\varphi_s(x)| \leqslant \delta$, for all $x > n_0$, and the function $g : (0,\infty) \to (0,\infty)$ satisfies that $\gamma(x) := \sum_{j=0}^{\infty}\prod_{i=0}^{j} 1/g(x+pi)$, is bound for all $x > n_0$, then there exists a unique solution $\varphi : (0,\infty) \to \mathbf{R}$ of Eq.* (3.20) *such that $|\varphi(x) - \varphi_s(x)| \leqslant \gamma(x)\delta$ for all $x > n_0$.*

Theorem 9. *If function* $\varphi_s:(0,\infty)\to\mathbf{R}$ *satisfies the inequality* $|\varphi_s(x+p)-g(x)\varphi_s(x)|\leqslant\phi(x)$ *for all* $x>n_0$ *and the functions* g *and* $\phi:(0,\infty)\to(0,\infty)$ *satisfy that* $\Phi(x):=\sum_{j=0}^{\infty}\phi(x+pj)\prod_{i=0}^{j}1/g(x+pi)<\infty$ *for all* $x\in(0,\infty)$, *then there exists a unique solution* $\varphi:(0,\infty)\to\mathbf{R}$ *of Eq.* (3.20) *such that* $|\varphi(x)-\varphi_s(x)|\leqslant\Phi(x)$ *for all* $x>n_0$.

Theorem 10. *If function* $\varphi_s:(0,\infty)\to(0,\infty)$ *satisfies the inequality*

$$\left|\frac{\varphi_s(x+p)}{g(x)\varphi_s(x)}-1\right|\leqslant\psi(x),\quad \forall x>n_0,$$

and the functions $g:(0,\infty)\to(0,\infty)$ *and* $\psi:(0,\infty)\to(0,1)$ *satisfy that* $\gamma(x):=\sum_{j=0}^{\infty}\prod_{i=0}^{j}1/g(x+pi)$ *is bound for all* $x>n_0$ *and that both* $\alpha(x):=\sum_{i=0}^{\infty}\log(1-\psi(x+pi))$ *and* $\beta(x):=\sum_{i=0}^{\infty}\log(1+\psi(x+pi))$ *are bounded for all* $x>n_0$, *then there exists a unique solution* $\varphi:(0,\infty)\to(0,\infty)$ *of Eq.* (3.20) *such that* $e^{\alpha(x)}\leqslant\varphi(x)/\varphi_s(x)\leqslant e^{\beta(x)}$ *for all* $x>n_0$.

In Theorems 8–10 the most important thing is convergence of series $\gamma(x)$, $\Phi(x)$, $\alpha(x)$ and $\beta(x)$ and it is discussed further in [17]. Observe the series $\sum_{j=0}^{\infty}\prod_{i=0}^{j}1/g(x+pi)$ in Theorem 8, for example, which is convergent if $\liminf_{k\to\infty}g(x+pk)>1$. However, the condition of the limit is weaker than the requirement for (3.16), i.e., $|g(x)|\geqslant r>1$, $\forall x\in(0,+\infty)$. In this sense, Baker's conditions are weakened in Jung's results and in Kim's. Recently, using the same method as in Jung's works and Kim's, Tiberiu [27] investigates the Hyers–Ulam–Rassias stability for the general linear equation (2.8) and the stability in the sense of Ger for Eq. (2.9). Let $\mathbf{K}$ be either the field $\mathbf{R}$ of real numbers or the field $\mathbf{C}$ of complex numbers, $\mathbf{R}_+$ the set of all nonnegative real numbers and $\mathbf{R}_+^*$ the set of all positive real numbers.

Theorem 11. *Let* X *be a Banach space over the field* $\mathbf{K}$ *and* S *a nonempty set. Let* $f:S\to S$, $h:S\to X$, $g:S\to\mathbf{K}\backslash\{0\}$ *and* $\psi:S\to\mathbf{R}_+$ *be given functions such that*

$$\Phi(x):=\sum_{k=0}^{\infty}\frac{\psi(f^k(x))}{\prod_{j=0}^{k}|g(f^j(x))|}<\infty \tag{3.21}$$

for all $x\in S$. *If a function* $\varphi_s:S\to X$ *satisfies*

$$\bigl\|\varphi_s\bigl(f(x)\bigr)-g(x)\varphi_s(x)-h(x)\bigr\|\leqslant\psi(x) \tag{3.22}$$

for all $x\in S$, *then there exists a unique function* $\varphi:S\to X$ *such that* φ *satisfies Eq.* (2.8) *and* $\|\varphi(x)-\varphi_s(x)\|\leqslant\Phi(x)$ *for all* $x\in S$.

Theorem 12. *Let* S *be a nonempty set and let* $f:S\to S$, $g:S\to\mathbf{R}_+^*$, *and* $\psi:S\to\,]0,1[$ *be given functions such that*

$$\alpha(x):=\prod_{j=0}^{+\infty}\bigl[1-\psi\bigl(f^j(x)\bigr)\bigr]>0\quad\text{and}\quad\beta(x):=\prod_{j=0}^{+\infty}\bigl[1+\psi\bigl(f^j(x)\bigr)\bigr]<\infty \tag{3.23}$$

for all $x\in S$. *If a function* $\varphi_s:S\to\mathbf{R}_+^*$ *satisfies*

$$\left|\frac{\varphi_s(f(x))}{g(x)\varphi_s(x)} - 1\right| \leqslant \psi(x) \tag{3.24}$$

for all $x \in S$, then there exists a unique function $\varphi : S \to \mathbf{R}_+^*$ such that φ satisfies Eq. (2.9) *and $\alpha(x) \leqslant \varphi(x)/\varphi_s(x) \leqslant \beta(x)$ for all $x \in S$.*

To be checked easily, condition (3.21) in Theorem 11 can be simplified as

$$\liminf_{j\to\infty} \frac{\psi(f^{j-1}(x))}{\psi(f^j(x))} \bigl|g\bigl(f^j(x)\bigr)\bigr| > 1, \quad \forall x \in S. \tag{3.25}$$

Similarly, condition (3.23) in Theorem 12 can be simplified as

$$\sum_{j=0}^{\infty} \psi\bigl(f^j(x)\bigr) < +\infty, \quad \forall x \in S. \tag{3.26}$$

4. On nonlinear equations

The Hyers–Ulam stability of the nonlinear functional equation

$$\varphi(x) = F\bigl(x, \varphi\bigl(f(x)\bigr)\bigr) \tag{4.27}$$

was discussed by Baker [1] in 1991.

Theorem 13. *Let S be a nonempty set and (X, d) be a complete metric space. Let $f : S \to S$, $F : S \times X \to X$ and $0 \leqslant \lambda < 1$. Suppose that*

$$d\bigl(F(x,u), F(x,v)\bigr) \leqslant \lambda d(u,v), \quad \forall x \in S,\ \forall u, v \in X, \tag{4.28}$$

and that $\varphi_s : S \to X$, $\delta > 0$ such that

$$d\bigl(\varphi_s(x), F\bigl(x, \varphi_s\bigl(f(x)\bigr)\bigr)\bigr) \leqslant \delta, \quad \forall x \in S. \tag{4.29}$$

Then there is a unique function $\varphi : S \to X$ such that $\varphi(x) = F(x, \varphi(f(x)))$ for all $x \in S$ and

$$d\bigl(\varphi(x), \varphi_s(x)\bigr) \leqslant \frac{\delta}{1-\lambda}, \quad \forall x \in S. \tag{4.30}$$

Proof. Let $Y = \{a : S \to X \mid \sup\{d(a(x), \varphi_s(x)) \mid x \in S\} < +\infty\}$. For $a, b \in Y$ define $\rho(a,b) = \sup\{d(a(x), b(x)) \mid x \in S\}$. Then $\varphi_s \in Y$, ρ is a metric on Y, and convergence with respect to ρ means uniform convergence on S with respect to d. Moreover, the completeness of X with respect to d implies the completeness of Y with respect to ρ.

For $a \in Y$ define $T(a) : S \to X$ by

$$T(a)(x) = F\bigl(x, a\bigl(f(x)\bigr)\bigr), \quad \forall x \in S.$$

Then T maps Y into Y. If $a, b \in Y$ then for all $x \in S$,

$$\begin{aligned} d\bigl(T(a)(x), T(b)(x)\bigr) &= d\bigl(F\bigl(x, a\bigl(f(x)\bigr)\bigr), F\bigl(x, b\bigl(f(x)\bigr)\bigr)\bigr) \\ &\leqslant \lambda d\bigl(a\bigl(f(x)\bigr), b\bigl(f(x)\bigr)\bigr) \leqslant \lambda \rho(a,b), \end{aligned} \tag{4.31}$$

by (4.28). Thus,

$$\rho\big(T(a), T(b)\big) \leqslant \lambda\rho(a, b), \quad \forall a, b \in Y.$$

According to the well-known proof of Banach's fixed point theorem, there exists a unique φ in Y such that $\varphi = T(\varphi)$ and

$$\begin{aligned}\rho(\varphi, \varphi_s) &\leqslant \rho\big(\varphi, T(\varphi_s)\big) + \rho\big(T(\varphi_s), \varphi_s\big)\\ &\leqslant \rho\big(T(\varphi), T(\varphi_s)\big) + \delta \leqslant \lambda\rho(\varphi, \varphi_s) + \delta,\end{aligned}$$

so that $\rho(\varphi, \varphi_s) \leqslant \delta/(1-\lambda)$. That is, there exists a unique solution φ of Eq. (4.27) such that inequality (4.30) hold. □

An interesting example of nonlinear equation is Böttcher's equation

$$\varphi\big(f(x)\big) = \varphi(x)^p \quad (p \neq 1) \tag{4.32}$$

which was proved by Brydak [4] to be iteratively stable if $p > 1$ is a given number and $f : I = (0, a) \to \mathbf{R}$ $(0 < a \leqslant \infty)$ is a continuous function such that $0 < f(x) < x$, $\forall x \in I$, and $\lim_{x\to 0} f(x)/x^p$ converges to a positive number. Applying Theorem 13, we can further give the Hyers–Ulam stability of the equation.

Theorem 14. *Suppose that* (i) $f : \mathbf{R} \to \mathbf{R}$, $p > 1$ *and* $\varphi_s : \mathbf{R} \to [1, +\infty)$ *satisfies*

$$\big|\varphi_s(x) - \varphi_s\big(f(x)\big)^{1/p}\big| \leqslant \delta, \quad \forall x \in \mathbf{R}, \tag{4.33}$$

for a constant $\delta > 0$, *or that* (ii) $f : \mathbf{R} \to \mathbf{R}$ *is homeomorphic*, $0 < p < 1$, *and* $\varphi_s : \mathbf{R} \to [1, +\infty)$ *satisfies*

$$\big|\varphi_s\big(f(x)\big) - \varphi_s(x)^p\big| \leqslant \delta, \quad \forall x \in \mathbf{R}, \tag{4.34}$$

for a constant $\delta > 0$. *Then there is a unique solution* $\varphi : \mathbf{R} \to [1, +\infty)$ *of Eq.* (4.32) *such that* $|\varphi(x) - \varphi_s(x)| \leqslant \max\{1/|p-1|, p/|p-1|\}\delta$ *for all* $x \in \mathbf{R}$.

Proof. In case $p > 1$, consider the equivalent form of Eq. (4.32)

$$\varphi(x) = \varphi\big(f(x)\big)^{1/p}. \tag{4.35}$$

Regard $[1, +\infty)$ as a complete metric space and let $F(x, u) = u^{1/p}$ where $x \in \mathbf{R}$, $u \geqslant 1$. Then F maps $\mathbf{R} \times [1, +\infty)$ into $[1, +\infty)$. By the mean value theorem,

$$\big|F(x, u) - F(x, v)\big| = |u^{1/p} - v^{1/p}| \leqslant \frac{1}{p}|u - v|, \quad \forall x \in \mathbf{R},\ \forall u, v \geqslant 1,$$

since $p > 1$. Thus, the Hyers–Ulam stability of (4.35) is implied by Theorem 13 and the result of part (i) is proved.

In case $0 < p < 1$, inequality (4.34) implies that

$$\big|\varphi_s(x) - \varphi_s\big(f^{-1}(x)\big)^p\big| \leqslant \delta, \quad \forall x \in \mathbf{R}, \tag{4.36}$$

since $f:\mathbf{R}\to\mathbf{R}$ is homeomorphic. Let $F(x,u)=u^p$ where $x\in\mathbf{R}$, $u\geqslant 1$. Then F maps $\mathbf{R}\times[1,+\infty)$ into $[1,+\infty)$. By the mean value theorem,

$$\bigl|F(x,u)-F(x,v)\bigr|=|u^p-v^p|\leqslant p|u-v|,\quad \forall x\in\mathbf{R},\ \forall u,v\geqslant 1,$$

since $0<p<1$. By inequality (4.36) and Theorem 13, there exists a unique solution $\varphi:\mathbf{R}\to[1,+\infty)$ of equation $\varphi(x)=\varphi(f^{-1}(x))^p$ such that $|\varphi(x)-\varphi_s(x)|\leqslant\delta/(1-p)$ for all $x\in\mathbf{R}$. That is, if $0<p<1$ and inequality (4.34) hold, there exists a unique solution $\varphi:\mathbf{R}\to[1,+\infty)$ of Eq. (4.32) satisfying inequality $|\varphi(x)-\varphi_s(x)|\leqslant\delta/(1-p)$ for all $x\in\mathbf{R}$. □

5. On iterative equations

Consider the iterative equation

$$G\bigl(\varphi(x),\varphi^2(x),\dots,\varphi^n(x)\bigr)=F(x),\tag{5.37}$$

i.e., a functional equation with iterates of the unknown function, where $x\in I=[a,b]$ and a,b are constants satisfies $a\leqslant b$, $F:I\to I$ is given, $\varphi:I\to I$ is unknown and φ^i denotes the ith iterate of φ, i.e., $\varphi^0(x)=x$ and $\varphi^{i+1}(x)=\varphi(\varphi^i(x))$ for all $x\in I$ and all $i=0,1,2,\dots$. Existence of continuous solutions is given in [25].

The Hyers–Ulam stability of Eq. (5.37) was discussed by Xu and Zhang [32] under the following hypotheses:

(H1) $G:I^n=I\times\cdots\times I\to I$ is continuous, $G(a,\dots,a)=a$, $G(b,\dots,b)=b$;

(H2) There exist constants $B_i\geqslant 0$ $(i=1,\dots,n)$ such that for $y_i,z_i\in I$ $(i=1,\dots,n)$

$$\bigl|G(y_1,\dots,y_n)-G(z_1,\dots,z_n)\bigr|\leqslant\sum_{i=1}^n B_i|y_i-z_i|;\tag{5.38}$$

(H3) There exist constants $C_1>0$, $C_i\geqslant 0$ $(i=2,\dots,n)$ such that

$$G(y_1,\dots,y_n)-G(z_1,\dots,z_n)\geqslant C_1(y_1-z_1)-\sum_{i=2}^n C_i|y_i-z_i|\tag{5.39}$$

for all $y_i,z_i\in I$, $i=1,\dots,n$, with $y_1\geqslant z_1$.

Theorem 15. *Suppose that hypotheses* (H1)–(H3) *hold and that $F:I\to I$ is a Lipschitzian mapping fixing the end-points of I with* $\mathrm{Lip}(F)\leqslant M_0$ *for positive constant M_0. If φ_s: $I\to I$ is a Lipschitzian mapping fixing end-points of I with* $\mathrm{Lip}(\varphi_s)\leqslant M$ *such that*

$$\bigl|F(x)-G\bigl(\varphi_s(x),\dots,\varphi_s^n(x)\bigr)\bigr|\leqslant\delta,\quad\forall x\in I,\tag{5.40}$$

for a constant $\delta>0$, then there exists a unique continuous solution $\varphi:I\to I$ of Eq. (5.37) *such that $|\varphi(x)-\varphi_s(x)|\leqslant\gamma\delta$ for all $x\in I$, where*

$$\gamma=\left(C_1-\sum_{i=2}^n C_iM^{i-1}-\max\left\{\sum_{i=2}^n C_i\sum_{j=0}^{i-2}M^j,\sum_{i=2}^n B_i\sum_{j=0}^{i-2}M^j\right\}\right)^{-1},$$

provided $C_1>\sum_{i=2}^n C_iM^{i-1}+\max\{M_0/M,\sum_{i=2}^n C_i\sum_{j=0}^{i-2}M^j,\sum_{i=2}^n B_i\sum_{j=0}^{i-2}M^j\}$.

This result is applicable to the polynomial-like iterative equation

$$\lambda_1\varphi(x) + \lambda_2\varphi^2(x) + \cdots + \lambda_n\varphi^n(x) = F(x), \tag{5.41}$$

studied in [33,34]. In order to cope with the equation where λ_j's are functions of x (i.e., the so-called problem of variable coefficients [36]), we will generalize Theorem 15 to the general form of iterative equations

$$G\big(x, \varphi(x), \varphi^2(x), \ldots, \varphi^n(x)\big) = F(x), \quad x \in I = [a, b]. \tag{5.42}$$

Existence of solutions is referred to [31,36].

Consider the same hypotheses (H1) and (H2) for $G : I^{n+1} \to I$, renamed by (H1′) and (H2′) respectively, and the hypothesis

(H3′) There exist constants $C_1 > 0$, $C_i \geqslant 0$, $i = 0, 2, 3, \ldots, n$, such that

$$G(y_0, y_1, \ldots, y_n) - G(z_0, z_1, \ldots, z_n) \geqslant C_1(y_1 - z_1) - \sum_{i=0,\, i\neq 1}^{n} C_i|y_i - z_i| \tag{5.43}$$

for $y_i, z_i \in I$ $(i = 0, 1, \ldots, n)$ with $y_1 \geqslant z_1$.

Let $\mathcal{C}(I)$ consist of all continuous functions on $I = [a, b]$, $\mathcal{C}_+(I) := \{\varphi \in \mathcal{C}(I){:}\ a = \varphi(a) \leqslant \varphi(x) \leqslant \varphi(b) = b\}$ and

$$\mathcal{C}(I; m, M) := \left\{\varphi \in \mathcal{C}(I){:}\ m \leqslant \frac{\varphi(x_2) - \varphi(x_1)}{x_2 - x_1} \leqslant M,\ \forall x_1, x_2 \in I,\ x_1 \neq x_2\right\},$$

where m, M are constants such that $m \leqslant M$. Let $\mathcal{C}_+(I; m, M) := \mathcal{C}(I; m, M) \cap \mathcal{C}_+(I)$.

Theorem 16. *Suppose that hypotheses* (H1′)–(H3′) *hold and that* $F \in \mathcal{C}_+(I; m_0, M_0)$, *where* $0 < m_0 \leqslant M_0$. *If* $\varphi_s \in \mathcal{C}_+(I; m, M)$, *where* $0 < m \leqslant M$, *such that*

$$\big|F(x) - G\big(x, \varphi_s(x), \varphi_s^2(x), \ldots, \varphi_s^n(x)\big)\big| \leqslant \delta, \quad \forall x \in I, \tag{5.44}$$

for a constant $\delta > 0$, *then there exists a unique continuous solution* $\varphi \in \mathcal{C}_+(I; m, M)$ *of Eq.* (5.42) *such that*

$$\big|\varphi(x) - \varphi_s(x)\big| \leqslant \gamma\delta, \quad \forall x \in I, \tag{5.45}$$

where

$$\gamma = \left(C_1 - \max\left\{\frac{C_0}{m} + \sum_{i=2}^{n} C_i \sum_{j=0}^{i-2} M^j, \frac{B_0}{m} + \sum_{i=2}^{n} B_i \sum_{j=0}^{i-2} M^j\right\} - \frac{C_0}{m} - \sum_{i=2}^{n} C_i M^{i-1}\right)^{-1},$$

provided

$$B_1 \leqslant \frac{m_0}{m} - \frac{B_0}{m} - \sum_{i=2}^{n} B_i M^{i-1}, \tag{5.46}$$

$$C_1 > \max\left\{\frac{M_0}{M}, \frac{C_0}{m} + \sum_{i=2}^{n} C_i \sum_{j=0}^{i-2} M^j, \frac{B_0}{m} + \sum_{i=2}^{n} B_i \sum_{j=0}^{i-2} M^j\right\} + \frac{C_0}{m} + \sum_{i=2}^{n} C_i M^{i-1}. \tag{5.47}$$

This theorem implies that Eq. (5.42) possesses Hyers–Ulam stability if constants B_j's and C_j's in (H2′) and (H3′) satisfy (5.46) and (5.47). Our requirements (H2′) and (H3′) are much weaker than the corresponding ones in [25] and [31]. Actually, ours (H2′) and (H3′) allow G not to be monotone. For example, $G(y_0, y_1, y_2) = (21/20)y_1 - (1/20)y_0 y_2^2$, $\forall y_0, y_1, y_2 \in I = [0, 1]$. More concretely, the iterative equation

$$\frac{21}{20}\varphi(x) - \frac{1}{20}x\big(\varphi^2(x)\big)^2 = F(x), \quad x \in I = [0, 1], \tag{5.48}$$

with $F(x) = (3/2)x$ as $0 \leqslant x \leqslant 1/3$ and $(3/4)x + 1/4$ as $1/3 < x \leqslant 1$, is Hyers–Ulam stable. In fact, take $G(y_0, y_1, y_2) = (21/20)y_1 - (1/20)y_0 y_2^2$. Obviously, $G(0, 0, 0) = 0$, $G(1, 1, 1) = 1$,

$$\big|G(y_0, y_1, y_2) - G(z_0, z_1, z_2)\big| \leqslant \frac{1}{20}|y_0 - z_0| + \frac{21}{20}|y_1 - z_1| + \frac{1}{10}|y_2 - z_2|$$

for $y_i, z_i \in I$, $i = 1, 2$, and for $y_1 \geqslant z_1$,

$$G(y_1, y_2) - G(z_1, z_2) \geqslant \frac{21}{20}(y_1 - z_1) - \frac{1}{20}|y_0 - z_0| - \frac{1}{10}|y_2 - z_2|,$$

which verify the hypotheses (H1′)–(H3′) with constants $B_0 = C_0 = 1/20$, $B_1 = C_1 = 21/20$ and $B_2 = C_2 = 1/10$. Consider the stability in the class $\mathcal{C}_+(I; 1/2, 2)$. Since $F \in \mathcal{C}_+(I; 3/4, 3/2)$, we can check that conditions (5.46) and (5.47) are satisfied. By Theorem 16, Eq. (5.48) is Hyers–Ulam stable in the class of functions.

It is worthy mentioning that the Hyers–Ulam stability is different concept from continuous dependence as discussed for Eqs. (5.37), (5.41) and (5.42) in [25,31,33,36]. Even though the stronger conditions in [25,31] corresponding to (H1′)–(H3′) hold, continuous dependence implies the difference between solutions φ_1, φ_2 in $\mathcal{C}_+(I; m, M)$ is arbitrarily small if the given functions F_1, F_2 are sufficiently close to each other in $\mathcal{C}_+(I; m_0, M_0)$. However, for the Hyers–Ulam stability, the approximate solution ϕ_s may not be a solution of Eq. (5.37) (or (5.42)) for a given function F_s in $\mathcal{C}_+(I; m_0, M_0)$ although $\phi_s \in \mathcal{C}_+(I; m, M)$. Theorem 16 implies that there exists a solution $\varphi \in \mathcal{C}_+(I; m, M)$ near φ_s in $\mathcal{C}_+(I; m, M)$ no matter whether φ_s satisfies $G(\varphi_s(x), \varphi_s^2(x), \ldots, \varphi_s^n(x)) \in \mathcal{C}_+(I; m_0, M_0)$ or not.

6. Proof of the generalized theorem

Lemma 1. *Suppose that $\varphi \in \mathcal{C}_+(I; m, M)$, where $0 < m \leqslant M$. If the reals C_j $(j = 0, 2, \dots, n)$ satisfy that $C_1 > C_0/m + \sum_{i=2}^{n} C_i M^{i-1}$, then $L\varphi$, defined by*

$$L\varphi(x) = G\bigl(\varphi^{-1}(x), x, \varphi(x), \dots, \varphi^{n-1}(x)\bigr), \tag{6.49}$$

is an orientation-preserving homeomorphism from I onto itself, and

$$(L\varphi)^{-1} \in \mathcal{C}_+\left(I; \frac{1}{B_1 + B_0/m + \sum_{i=2}^{n} B_i M^{i-1}}, \frac{1}{C_1 - C_0/m - \sum_{i=2}^{n} C_i M^{i-1}}\right). \tag{6.50}$$

Proof. Since $m > 0$ we see φ is invertible, $\varphi^{-1} \in \mathcal{C}_+(I; M^{-1}, m^{-1})$ and $\varphi^i \in \mathcal{C}_+(I; m^i, M^i)$ by lemmas in [36]. Then $L\varphi(a) = a$, $L\varphi(b) = b$ by hypothesis (H1′). Let

$$\xi := C_1 - \frac{C_0}{m} - \sum_{i=2}^{n} C_i M^{i-1}, \qquad \tilde{\xi} := B_1 + \frac{B_0}{m} + \sum_{i=2}^{n} B_i M^{i-1}. \tag{6.51}$$

Obviously, $\xi \leqslant \tilde{\xi}$ because for any $y_1', y_1'', y_i, \in I$ $(i = 0, 2, \dots, n)$ with $y_1' > y_1''$,

$$C_1(y_1' - y_1'') \leqslant G(y_0, y_1', y_2, \dots, y_n) - G(y_0, y_1'', y_2, \dots, y_n) \leqslant B_1(y_1' - y_1'')$$

by (H3′) and (H2′), which implies $C_1 \leqslant B_1$. Then, for any $x_1, x_2 \in I$ with $x_2 > x_1$,

$$\begin{aligned} &L\varphi(x_2) - L\varphi(x_1) \\ &\quad \geqslant C_1(x_2 - x_1) - C_0\bigl|\varphi^{-1}(x_2) - \varphi^{-1}(x_1)\bigr| - \sum_{i=2}^{n} C_i\bigl|\varphi^{i-1}(x_2) - \varphi^{i-1}(x_1)\bigr| \\ &\quad \geqslant C_1(x_2 - x_1) - \frac{C_0}{m}(x_2 - x_1) - \sum_{i=2}^{n} C_i M^{i-1}(x_2 - x_1) \\ &\quad \geqslant \xi(x_2 - x_1) > 0 \end{aligned} \tag{6.52}$$

because of (5.47). Similarly,

$$\begin{aligned} &L\varphi(x_2) - L\varphi(x_1) \\ &\quad \leqslant B_1(x_2 - x_1) + B_0\bigl|\varphi^{-1}(x_2) - \varphi^{-1}(x_1)\bigr| + \sum_{i=2}^{n} B_i\bigl|\varphi^{i-1}(x_2) - \varphi^{i-1}(x_1)\bigr| \\ &\quad \leqslant B_1(x_2 - x_1) + \frac{B_0}{m}(x_2 - x_1) + \sum_{i=2}^{n} B_i M^{i-1}(x_2 - x_1) \\ &\quad \leqslant \tilde{\xi}(x_2 - x_1). \end{aligned} \tag{6.53}$$

Hence $L\varphi \in \mathcal{C}_+(I; \xi, \tilde{\xi})$, and $L\varphi$ is an orientation-preserving homeomorphism from I onto itself. Obviously, $(L\varphi)^{-1} \in \mathcal{C}_+(I; \tilde{\xi}^{-1}, \xi^{-1})$. □

Lemma 2. *Suppose that* $0 < m \leqslant M$*,* $0 < m_0 \leqslant M_0$ *and* $F \in \mathcal{C}_+(I; m_0, M_0)$*, and suppose that the reals* B_j *and* C_j $(j = 0, 1, \ldots, n)$ *satisfy that* $B_1 \leqslant m_0/m - B_0/m - \sum_{i=2}^{n} B_i M^{i-1}$ *and* $C_1 \geqslant M_0/M + C_0/m + \sum_{i=2}^{n} C_i M^{i-1}$*. Then for each* $\varphi_0 \in \mathcal{C}_+(I; m, M)$ *the sequence* (φ_k) *is well defined by*

$$\varphi_k := (L\varphi_{k-1})^{-1} \circ F \tag{6.54}$$

and

$$L\varphi_{k-1}(x) := G\big(\varphi_{k-1}^{-1}(x), x, \varphi_{k-1}(x), \ldots, \varphi_{k-1}^{n-1}(x)\big) \tag{6.55}$$

and satisfies $\varphi_k \in \mathcal{C}_+(I; m, M)$*,* $k = 1, 2, \ldots.$

Proof. Let $L\varphi_0(x) := G(\varphi_0^{-1}(x), x, \varphi_0(x), \ldots, \varphi_0^{n-1}(x))$. By Lemma 1, $L\varphi_0(x)$ is well-defined mapping I onto itself homeomorphically, and satisfies $(L\varphi_0)^{-1} \in \mathcal{C}_+(I; \tilde{\xi}^{-1}, \xi^{-1})$, where $\xi, \tilde{\xi}$ are defined in (6.51). Thus $\varphi_1(x) := (L\varphi_0)^{-1} \circ F(x)$ is meaningful. Moreover, $\varphi_1 \in \mathcal{C}_+(I; m_0/\tilde{\xi}, M_0/\xi) \subset \mathcal{C}_+(I; m, M)$ by lemmas in [36] and the assumptions (5.46) and (5.47) on B_j, C_j, $j = 0, 1, \ldots, n$. Further assume that results in Lemma 2 are true for the integer k. By Lemma 1, $L\varphi_k(x) := G(\varphi_k^{-1}(x), x, \varphi_k(x), \ldots, \varphi_k^{n-1}(x))$ is also well-defined mapping I onto itself homeomorphically, and satisfies $(L\varphi_k)^{-1} \in \mathcal{C}_+(I; \tilde{\xi}^{-1}, \xi^{-1})$. Thus $\varphi_{k+1}(x) := (L\varphi_k)^{-1} \circ F(x)$ is meaningful and $\varphi_{k+1} \in \mathcal{C}_+(I; m_0/\tilde{\xi}, M_0/\xi) \subset \mathcal{C}_+(I; m, M)$. Thus, Lemma 2 is proved by induction. □

Proof of Theorem 16. For simplicity, let $\xi, \tilde{\xi}$ remain as in (6.51) and

$$\eta := \max\left\{\frac{C_0}{m} + \sum_{i=2}^{n} C_i \sum_{j=0}^{i-2} M^j, \frac{B_0}{m} + \sum_{i=2}^{n} B_i \sum_{j=0}^{i-2} M^j\right\}. \tag{6.56}$$

Take $\varphi_0 = \varphi_s$. Then a sequence (φ_k) is constructed as in Lemma 2 successively. Lemmas 1 and 2 imply that $\varphi_k \in \mathcal{C}_+(I; m, M)$, $L\varphi_k$ is an orientation-preserving homeomorphism from I onto itself and $(L\varphi_k)^{-1} \in \mathcal{C}_+(I; \tilde{\xi}^{-1}, \xi^{-1})$. Now we claim

$$\|F - L\varphi_{k-1} \circ \varphi_{k-1}\| \leqslant \left(\frac{\eta}{\xi}\right)^{k-1} \delta, \tag{6.57}$$

$$\|\varphi_k - \varphi_{k-1}\| \leqslant \frac{1}{\xi}\left(\frac{\eta}{\xi}\right)^{k-1} \delta, \tag{6.58}$$

where $\|\cdot\|$ denotes the supremum norm on I.

Inequalities (6.57) and (6.58) are obvious for $k = 1$. Assume they are true for the integer k. By lemmas in [35] (or [36]) and (H3′),

$$\begin{aligned} F(x) - L\varphi_k \circ \varphi_k(x) &= G\big(\varphi_{k-1}^{-1} \circ \varphi_k(x), \varphi_k(x), \varphi_{k-1} \circ \varphi_k(x), \ldots, \varphi_{k-1}^{n-1} \circ \varphi_k(x)\big) \\ &\quad - G\big(x, \varphi_k(x), \varphi_k^2(x), \ldots, \varphi_k^n(x)\big) \\ &\geqslant -C_0 \big\|\varphi_{k-1}^{-1} - \varphi_k^{-1}\big\| - \sum_{i=2}^{n} C_i \big\|\varphi_{k-1}^{i-1} - \varphi_k^{i-1}\big\| \end{aligned}$$

$$\geqslant -\left(\frac{C_0}{m} + \sum_{i=2}^{n} C_i \sum_{j=0}^{i-2} M^j\right)\|\varphi_{k-1} - \varphi_k\|. \tag{6.59}$$

Similarly, by (H2′),

$$\begin{aligned} F(x) - L\varphi_k \circ \varphi_k(x) &\leqslant B_0 \left\|\varphi_{k-1}^{-1} - \varphi_k^{-1}\right\| + \sum_{i=2}^{n} B_i \left\|\varphi_{k-1}^{i-1} - \varphi_k^{i-1}\right\| \\ &\leqslant \left(\frac{B_0}{m} + \sum_{i=2}^{n} B_i \sum_{j=0}^{i-2} M^j\right)\|\varphi_{k-1} - \varphi_k\|. \end{aligned} \tag{6.60}$$

It follows that

$$\begin{aligned} &\|F - L\varphi_k \circ \varphi_k\| \\ &\quad\leqslant \max\left\{\frac{C_0}{m} + \sum_{i=2}^{n} C_i \sum_{j=0}^{i-2} M^j, \frac{B_0}{m} + \sum_{i=2}^{n} B_i \sum_{j=0}^{i-2} M^j\right\}\|\varphi_{k-1} - \varphi_k\| \\ &\quad\leqslant \eta\left(\frac{1}{\xi}\left(\frac{\eta}{\xi}\right)^{k-1}\delta\right) = \left(\frac{\eta}{\xi}\right)^{k}\delta, \end{aligned} \tag{6.61}$$

by (6.58). Moreover,

$$\begin{aligned} \|\varphi_{k+1} - \varphi_k\| &= \left\|(L\varphi_k)^{-1} \circ F - (L\varphi_k)^{-1} \circ (L\varphi_k) \circ \varphi_k\right\| \\ &\leqslant \frac{1}{\xi}\left\|F - (L\varphi_k) \circ \varphi_k\right\| \leqslant \frac{1}{\xi}\left(\frac{\eta}{\xi}\right)^{k}\delta, \end{aligned} \tag{6.62}$$

by (6.61). Thus (6.57) and (6.58) are proved by induction.

For any positive integers k and l with $k > l$,

$$\begin{aligned} \|\varphi_k - \varphi_l\| &\leqslant \|\varphi_k - \varphi_{k-1}\| + \|\varphi_{k-1} - \varphi_{k-2}\| + \cdots + \|\varphi_{l+1} - \varphi_l\| \\ &\leqslant \frac{1}{\xi}\left(\frac{\eta}{\xi}\right)^{k-1}\delta + \frac{1}{\xi}\left(\frac{\eta}{\xi}\right)^{k-2}\delta + \cdots + \frac{1}{\xi}\left(\frac{\eta}{\xi}\right)^{l}\delta \\ &= \left(\frac{\delta}{\xi}\right)\frac{(\eta/\xi)^l - (\eta/\xi)^k}{1 - (\eta/\xi)}, \end{aligned} \tag{6.63}$$

by (6.58), where $\xi > \eta$ by (5.47). It follows that $\|\varphi_k - \varphi_l\| \to 0$ as $k, l \to +\infty$. Therefore, being a Cauchy sequence, $\{\varphi_k\}$ converges uniformly in the Banach space $\mathcal{C}(I)$. Let $\varphi = \lim_{k\to+\infty} \varphi_k$. Clearly, $\varphi \in \mathcal{C}_+(I; m, M)$. Furthermore, by (6.57),

$$\|F - L\varphi \circ \varphi\| = \lim_{k\to+\infty} \|F - L\varphi_k \circ \varphi_k\| \leqslant \lim_{k\to+\infty}\left(\frac{\eta}{\xi}\right)^{k}\delta = 0, \tag{6.64}$$

i.e., φ is a solution of Eq. (5.42). By (6.58),

$$\begin{aligned} \|\varphi - \varphi_s\| &= \lim_{k\to+\infty} \|\varphi_k - \varphi_0\| \\ &\leqslant \lim_{k\to+\infty}\left\{\|\varphi_k - \varphi_{k-1}\| + |\varphi_{k-1} - \varphi_{k-2}\| + \cdots + |\varphi_1 - \varphi_0\|\right\} \end{aligned}$$

$$\leqslant \lim_{k\to+\infty}\left\{\frac{1}{\xi}\left(\frac{\eta}{\xi}\right)^{k-1}\delta+\frac{1}{\xi}\left(\frac{\eta}{\xi}\right)^{k-2}\delta+\cdots+\frac{1}{\xi}\delta\right\}=\frac{1}{\xi-\eta}\delta. \tag{6.65}$$

This proves (5.45).

For an indirect proof of uniqueness, we assume that there exists another solution ϕ ($\neq \varphi$) of Eq. (5.42) in $\mathcal{C}_+(I; m, M)$ such that $|\phi(x) - \varphi_s(x)| \leqslant \varepsilon$, where ε is a positive constant and only depends on δ. Then

$$\begin{aligned}
\|\varphi-\phi\| &= \left\|(L\varphi)^{-1}\circ F-(L\phi)^{-1}\circ F\right\| \\
&\leqslant \xi^{-1}\|L\varphi-L\phi\| \leqslant \xi^{-1}\sum_{i=0}^{n}B_i\|\varphi^{i-1}-\phi^{i-1}\| \\
&\leqslant \xi^{-1}\left(\frac{B_0}{m}+\sum_{i=2}^{n}B_i\sum_{j=0}^{i-2}M^j\right)\|\varphi-\phi\|,
\end{aligned}$$

by Lemma 1 and (H2′), where lemmas in [35] (or [36]) are applied. It follows that

$$\left(1-\left(C_1-\frac{C_0}{m}-\sum_{i=2}^{n}C_iM^{i-1}\right)^{-1}\left(\frac{B_0}{m}+\sum_{i=2}^{n}B_i\sum_{j=0}^{i-2}M^j\right)\right)\|\varphi-\phi\|\leqslant 0.$$

It implies a contradiction that $\varphi \equiv \phi$ by (5.47). □

7. Some remarks

Being a subject in the field of functional equations, Hyers–Ulam stability is also applicable to differential equations, although few results (see [21,22]) are given for differential equations. Consider the differential equation

$$\frac{d\varphi(x)}{dx}=f\bigl(x,\varphi(x)\bigr) \tag{7.66}$$

in general. The Hyers–Ulam stability of (7.66) means that if φ_s is a δ-approximate solution, i.e., $|d\varphi_s(x)/dx - f(x,\varphi_s(x))| \leqslant \delta$ for all $x \in I$, then there exists a solution φ of (7.66) such that $|\varphi(x) - \varphi_s(x)| \leqslant \varepsilon$, where $\varepsilon > 0$ depends only on δ. It is related to but weaker than the concept of robust stability. The latter one concerns behaviors of Eq. (7.66) near a given solution φ. Without loss of generality we suppose that the solution $\varphi \equiv 0$ and $f(x,0) \equiv 0$. We say the solution $\varphi \equiv 0$ of Eq. (7.66) is *robust* if for any small $\varepsilon > 0$ there exist constants $\delta_1(\varepsilon) > 0$, $\delta_2(\varepsilon) > 0$ such that, for continuous function $g(x, y)$ and initial data y_0 with $\|g\| < \delta_1$ and $|y_0| \leqslant \delta_2$, the solution $y(x, x_0, y_0)$ of the disturbed equation

$$\frac{dy}{dx}=f(x,y)+g(x,y) \tag{7.67}$$

satisfies $|y(x, x_0, y_0)| < \varepsilon$ for all $x \geqslant x_0$.

For the differential equation

$$\varphi'(x)=\lambda\varphi(x), \tag{7.68}$$

the real case and complex case are investigated by Miura et al. in [21] and [22], respectively. Consider a differentiable map φ_s from an open interval I to a uniformly closed linear subspace A of $C(X)$, the Banach space of all complex-valued bounded continuous functions (equipped with the supremum norm $\|\cdot\|_\infty$) on a topological space X. Let ϵ be a nonnegative real number, and λ a complex number with $\operatorname{Re}\lambda \neq 0$. It is proved in [22] that φ_s can be approximated by the solution φ to the A-valued differential equation (7.68) if $\|\varphi_s'(x) - \lambda\varphi_s(x)\|_\infty \leqslant \epsilon$ holds for every $x \in I$.

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